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CANONICAL POLARIZATIONS OF PICARD SCHEMES*

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Introduction

In [2], [3] and [4] Hayashida and Nishi show the existence of curves of genus two with Jacobian varieties which are isomorphic to a product of elliptic curves. Actually, such curves are quite common. Lange [7, §2] shows that they are dense with respect to the Zariski topology on $\mathcal{M}_2 \times_{\mathbb{Z}} k$, where $\mathcal{M}_2$ denotes the coarse moduli scheme over $\mathbb{Z}$ of irreducible non-singular curves of genus two and k is an algebraically closed field. In case $k = \mathbb{C}$ he shows that such curves are even dense with respect to the Euclidean topology on $\mathcal{M}_2 \times_{\mathbb{Z}} k$. Lange also proves that the curves of genus three with Jacobian varieties isomorphic to a product of elliptic curves are dense in $\mathcal{M}_3 \times_{\mathbb{Z}} k$.

By studying Riemann matrices Martens [9, §4] shows that any product of isomorphic elliptic curves defined over $\mathbb{C}$ admits infinitely many automorphisms. He concludes that the Jacobian variety, $J(C)$, of a curve C carries infinitely many principal polarizations arising from canonical embeddings of C whenever $J(C)$ is isomorphic to a product of an elliptic curve with itself. Here we show the existence of curves of genus two and three with Jacobian an *elementary* abelian variety having infinitely many automorphisms. Such $J(C)$ also carry infinitely many principal polarizations arising from canonical embeddings of C. The purpose of this note is to show that the existence of such curves actually reflects a general phenomenon. We prove the following theorem.

Theorem. *Let $g = 2, 3$, p be a prime number, and R denote the ring of integers in $\bar{\mathbb{Q}}_p$. There exist smooth curves $\mathcal{C}$ over R of genus g such that $\mathrm{Pic}^0_{\mathcal{C}/R}$ has infinitely many R-automorphisms and such that the fibers of $\mathrm{Pic}^0_{\mathcal{C}/R}$ are elementary abelian varieties. Furthermore, $\mathrm{Pic}^0_{\mathcal{C}/R}$ carries infinitely many canonical polarizations.*

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To obtain curves C defined over $\mathbb{C}$ such that $J(C)$ is an elementary abelian variety having infinitely many automorphisms it suffices to choose a homomorphism $\phi : R \to \mathbb{C}$ and let $C = \mathscr{C} \times_{\phi} \mathbb{C}$.

1.

Let k be a field. Recall that an abelian variety A of dimension g defined over k is said to be k-elementary if A contains no nontrivial abelian subvarieties defined over k. In case k is of characteristic $p, p>0, A$ is said to be ordinary if there are p^g points of A of order dividing p with values in $\bar{k}$, or equivalently, if the p-divisible group $T_p(A)$ is isogenous to $gG_{1,0}$ over $\bar{k}$, [8].

Proposition 1. *Let k be an algebraic closure of a finite field and g be a positive integer, then there exists a k-elementary ordinary abelian variety A^0 of dimension g with a principal polarization λ^0 which is defined over k. Furthermore,* $\mathrm{End}_k A^0$ *is an order in a totally imaginary field Φ with $[\Phi : \mathbb{Q}] = 2g$.*

Proof. Lenstra and Oort [5] have shown that for each g there exist k-elementary ordinary abelian varieties. For such an abelian variety A, $\mathrm{End}_k A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a totally imaginary field Φ with $[\Phi : \mathbb{Q}] = 2g$, [17, Proposition 7.1, Theorem 7.2], [16, Theorem 2c]. A is k-isogenous to a principally polarized abelian variety A^0, [12, Corollary, p. 234], which is necessarily also k-elementary and ordinary.

Corollary. *Let $H^0 = \mathrm{Aut}_k A^0$, then H^0 is a finitely generated abelian group of rank $g-1$.*

Proof. By Dirichlet's theorem the group of units in $\mathrm{End}_k A^0$ has rank $g-1$.

Let k, A^0 and λ^0 be as in Proposition 1 and W denote the ring of Witt vectors of k. Let $\tilde{A}$ be the Serre–Tate lifting of A^0 to W [1, §3], [11, Ch. V, §3]; $\tilde{A}$ is a projective abelian scheme over W and the canonical map γ: $\mathrm{End}_W \tilde{A} \to \mathrm{End}_k A$ is a ring isomorphism. Let A^{0t} be the dual abelian variety of A^0, then the Serre–Tate lifting of A^{0t} is the dual abelian scheme $\tilde{A}^t$ of $\tilde{A}$. The principal polarization $\lambda^0 : A \to A^{0t}$ lifts to an isomorphism $\tilde{\lambda} : \tilde{A} \to \tilde{A}^t$ which is a polarization in the sense of [13, Ch. 6, §2].

Proposition 2. *Let W_0 be the quotient field of W and L be a finite algebraic extension of W_0, then there are canonical isomorphisms* $\mathrm{End}_W \tilde{A} \cong \mathrm{End}_L(\tilde{A} \times_W L) \cong \mathrm{End}_k A^0$. *In particular, $\tilde{A} \times_W L$ is L-elementary, and* $\mathrm{Aut}_W \tilde{A} \cong \mathrm{Aut}_L(\tilde{A} \times_W L) \cong \mathrm{Aut}_k A^0$ *are finitely generated abelian groups of rank $r-1$.*

Proof. There exist ring homomorphisms α and β such that

$$\begin{tikzcd}
 & \mathrm{End}_L(\tilde{A} \times L) \arrow[d, "\beta"] \\
\mathrm{End}_W \tilde{A} \arrow[ur, "\alpha"] \arrow[r, "\gamma"'] & \mathrm{End}_k A^0
\end{tikzcd}$$

commutes. By [15, §11.1, Proposition 12] β is injective; since γ is an isomorphism, β and α are also. The last assertion now follows from the Corollary of Proposition 1.

Let $\tilde{\mathscr{P}}$ (resp. $\mathscr{P}_L$, $\mathscr{P}^0$) denote the set of polarizations of $\tilde{A}$ (resp. $\tilde{A} \times_W L$, A^0). The group $H^0 = \mathrm{Aut}_k A^0$ acts on $\mathscr{P}^0$ and hence on $\tilde{\mathscr{P}}$ and on $\mathscr{P}_L$.

Corollary. *Let $h \in H^0$ be an element of infinite order, $\lambda^0 \in \mathscr{P}^0$ and $\tilde{\lambda} \in \tilde{\mathscr{P}}$ be as above, and let $\lambda_L = \tilde{\lambda} \times_W L \in \mathscr{P}_L$. Then $h^i \cdot \lambda^0 \neq \lambda^0$, $h^i \cdot \lambda_L \neq \lambda_L$, and hence $h^i \cdot \tilde{\lambda} \neq \tilde{\lambda}$, for all $i \in \mathbb{Z}$, $i \neq 0$.*

Proof. The corollary follows immediately from the fact that the group of automorphisms of a polarized abelian variety is finite, [6, VII, §2, Proposition 8].

2.

Definition [14]. Let S be a scheme and $g \geq 2$. A good curve of genus g over S is a proper, flat S-scheme $\mathscr{C}$ with geometric fibers $\mathscr{C}_s$ which are reduced, connected, one-dimensional schemes such that

(i) $\mathscr{C}_s$ contains no non-singular rational components.

(ii) $\mathrm{Pic}^0_{\mathscr{C}_s/k(s)}$ is an abelian variety of dimension g.

In particular, each component of $\mathscr{C}_s$ is non-singular.

Theorem. *Let k be the algebraic closure of a finite field, W be the ring of Witt vectors of k, W_0 be the quotient field of W, $(\tilde{A}, \tilde{\lambda})$ be the polarized abelian scheme of Section 1, and $g = 2, 3$. Then there exists a finite algebraic extension L of W_0 and a smooth curve $\mathscr{C}$ over the valuation ring R_L of L such that $A \times_W R_L$ and $\mathrm{Pic}^0_{\mathscr{C}/R_L}$ are R_L-isomorphic and such that the principally polarized abelian variety $(\tilde{A} \times_W L, \tilde{\lambda} \times_W L)$ (respectively, $(A \times_W k, \tilde{\lambda} \times_W k)$) is isomorphic to the canonically polarized $J(\mathscr{C} \times_W L) = \mathrm{Pic}^0_{\mathscr{C} \times_W L/L}$ (respectively, $J(\mathscr{C} \times_W k) = \mathrm{Pic}^0_{\mathscr{C} \times_W k/k}$).*

Proof. By the theorem of [14] (or, for the case $g = 2$, [19, Satz 2]) there exists a finite algebraic extension L of W_0 and a good curve C over L such that $\mathrm{Pic}^0_{C/L}$ with the canonical polarization is isomorphic to $(\tilde{A} \times_W L, \tilde{\lambda} \times_W L)$. Hence by [14, Lemma 6] there exists a good curve $\mathscr{C}$ over R_L having the properties stated. To show that $\mathscr{C}$ is smooth it suffices to show that each geometric fiber is irreducible. However, $\tilde{A} \times_{W'} k \cong A^0$ is k-elementary by construction and $\tilde{A} \times_{W'} L'$ is L'-elementary for any finite algebraic extension L' of L by Proposition 2, so each geometric fiber of $\mathscr{C}$ has only one component.

Definition. Let R be a discrete valuation ring with quotient field L and residue field k. Let $\mathscr{C}$ be a smooth curve over R. A polarization $\mu : \mathrm{Pic}^0_{\mathscr{C}/R} \to \mathrm{Pic}^{0\,\mathrm{t}}_{\mathscr{C}/R}$ in the sense of [13, Ch. 6, §2] is said to be canonical if $\mu \times L : J(\mathscr{C} \times_R L) \to J(\mathscr{C} \times_R L)^{\mathrm{t}}$ and $\mu \times k : J(\mathscr{C} \times_R k) \to J(\mathscr{C} \times_R k)^{\mathrm{t}}$ are both canonical polarizations.

Corollary. *Let $\mathscr{C}$ and R_L be as in the Theorem, then $\mathrm{Pic}^0_{\mathscr{C}/R_L}$ has infinitely many R_L-automorphisms and hence carries infinitely many canonical polarizations.*

Proof. By Proposition 2 and its Corollary $A \times_{W'} R_L \cong \mathrm{Pic}^0_{\mathscr{C}/R_L}$ has an R_L-automorphism h of infinite order and the polarizations $h^i \cdot \tilde{\lambda}$ and $h^j \cdot \tilde{\lambda}$ of $A \times_{W'} R_L$ are distinct for $i, j \in \mathbb{Z}$, $i \neq j$. Actually these polarizations are all canonical; to show this it suffices to prove that the polarizations $h \cdot \lambda_L$ and $h \cdot \lambda^0$ defined in the Corollary to Proposition 2) of $J(\mathscr{C} \times_{R_L} L)$ and of $J(\mathscr{C} \times_{R_L} k)$, respectively, are canonical. Let $C_L = \mathscr{C} \times_{R_L} L$, by the Theorem λ_L arises from a positive divisor Θ on $J(C_L)$, uniquely determined up to translation, [19, §§1,2]. Let $\phi : C_L \to J(C_L)$ be a canonical embedding, then $\deg \Theta^g = g!$ and Θ^{g-1} is numerically equivalent to $(g-1)!\phi(C_L)$, [10, §3, Proposition 3]. Let $h^{-1}\Theta$ denote the inverse image of the cycle Θ, [18, VIII, §4], then $\deg(h^{-1}\Theta)^g = g!$ and $(h^{-1}\Theta)^{g-1}$ is numerically equivalent to $(g-1)!h^{-1}(\phi(C_L))$, [18, VI, §3, Theorem 10], so by [10, §3, Theorem 3], $h^{-1}\Theta$ defines a canonical polarization of $J(C_L)$. The same argument shows that $h \cdot \lambda^0$ is a canonical polarization of $J(\mathscr{C} \times_{R_L} k)$.

Remark. The theorem stated in the introduction follows immediately from the Theorem and Corollary above, from Proposition 2, and from the construction of A^0.

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